Contributions of Riemann invariants to the Entropy of Extremal Black Holes.

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Abstract: We use the entropy function formalism introduced by A. Sen to obtain the entropy of $AdS_2 \times S^{d-2}$ extremal and static black holes in four and five dimensions, with higher derivative terms of a general type. Starting from a generalized Einstein–Maxwell action with nonzero cosmological constant, we examine all possible scalar invariants that can be formed from the complete set of Riemann invariants (up to order 10 in derivatives). The resulting entropies show the deviation from the well known Bekenstein–Hawking area law $S = A/4G$ for Einstein's gravity up to second order derivatives.

Keywords: Black Holes in String Theory, Black Holes, AdS-CFT Correspondence.

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1. Introduction

The idea of the "attractor mechanism" was firstly applied in the context of supergravity. Later it was reinterpreted to calculate the entropy of extremal black holes (BH)[1]. Many articles already have been published in which this method has been used successfully in different contexts [2],[3],[4]. The popularity of this idea possibly follows from the simplicity of the entropy function method. The fact that the geometry near the horizon implies that all scalar fields and sources take constant values, is perhaps, among its properties, what makes this method easier to apply. This idea is consistent with the so–called "attractor mechanics" for supersymmetric backgrounds, in which BH configurations near the horizon depend only on the electric and magnetic charge carried by the BH, and not on the asymptotic values of the corresponding scalar field [1]. Indeed, by using the entropy function mechanism, the entropy in the proximity of the BH horizon can be regarded as the Legendre transformation of specific suitable parameters. Since the entropy will depend on these charges, it is important to emphasize that the entropy function formalism computes the entropy of an extremal BH as the entropy of a non–extremal BH in the limit as we approach the horizon.

In d dimensions and near the horizon, the geometry of a spherically symmetric and extremal BH implies that the isometry group has the $SO(1,2) \times SO(d-1)$ form [5],[6],[7]. This follows from the fact that near the horizon the metric has the Robinson–Bertotti(RB) form $AdS_2 \times S^{d-2}$ [8],[9]. If we assume that the addition of higher derivative terms in

the action do not destroy the RB geometry near the horizon, then we could consider this metric as valid in general, even for dimensions larger than four. It is worth noting that the RB solution is not asymptotically flat, as it is the product of two spaces [10]: the anti-de Sitter AdS space and the sphere S. In $d = 4$ both are two–dimensional spaces with the same curvature constants, $2/R^2_{AdS}$, but with different sign, with the AdS space having negative curvature and being invariant with respect to the isometry group $SO(1, 2)$, while S^2 is the two–dimensional sphere with positive curvature and invariant with respect to the group $SO(3)$. Since the RB group of symmetry is bigger than the isometry group $SO(1, 1) \times SO(3)$ of the extremal Reissner Nordstrom (RN), the consequent symmetry increase makes it possible to consider the RB space as a vacuum state for this theory, similar to a Minkowski vacuum [11]. In this sense, the extremal RN black hole at infinity is asymptotically flat (*i.e.* Minkowski), but near the horizon the geometry is described by the RB metric. Therefore, the extremal RN BH can be seen as a gravitational soliton. This interpretation is closely related to the recent discovery of AdS/CFT correspondence, in which the AdS gravity can be linked with gauge theories defined on the AdS boundary.

In this paper we obtain the entropy of extremal and static BH taking into account higher derivative terms. The inclusion of these terms in theories of high order gravity can be done, for example, using the terms of Gauss-Bonnet(GB) [14],[15],[16]. These terms could also appear in various situations, such as string theories, branes[17],[18],[19] and semiclassical quantum gravity. In particular, higher derivative terms appear in string theories when the effective low-energy limit is evaluated. It is well-known that in higher order gravity problems associated with renormalization may occur. In the particular case of GB gravity, the action formed by three second order invariants insure the elimination of these renormalization problems, though other problems come up, such as the emergence of ghost fields (*i.e.* massive particle with spin two). The discussion of these problems is beyond the scope of this article, though a good review on these issues can be found in [12] and [13].

The main objective of this paper is to present a set of approximate solutions for the entropy of extremal BH's near the horizon of geometry in $d = 4$ and $d = 5$ dimensions. To this end we used Sen's mechanism, the initial condition derived from the Einstein–Maxwell theory with cosmological constant, as well as a complete set of Riemann invariants. It is important to point out that in the cases when the equations of motion near the BH horizon cannot be solved analytically, we can attempt to construct an iterative power series in the w parameter inversely to the BH charge. On the other hand, in order to include the Riemann set of invariants as corrections of higher derivatives, we have to considere the complete set of invariants defined by Carminati and McLenaghan (CM) [20], plus the m_6 invariant introduced by Zakhary and McIntosh [21].

The CM invariants are scalars built from the Riemann tensor R_{abcd}, the Weyl tensor C_{abcd} (and its dual) and the trace-free Ricci tensor, defined as $S_{ab} = R_{ab} - (1/d)R\, g_{ab}$ (in d-dimensions), leading to six real scalars $R, r_1, r_2, r_3, m_3, m_4$ and five complex scalars w_1, w_2, m_1, m_2, m_5, making a total of sixteen invariant scalars (we notice that r, w and m are respectively associated with Riemann, Weyl and mixed invariants). The CM set of scalars yields the required number of invariants for the Einstein-Maxwell and perfect

fluid cases. Furthermore, by including the m_6 invariant, it has been proven that the CM set becomes a *complete set*, as it covers the 90 possible cases (6 Petrov types $\times$ 15 Segre types)[21]. In other words, a complete set of invariants should contain (at least for $d = 4$ spacetimes) the already well-known physical invariants, as well as the geometric ones.

In Table 1 (see Appendix) we show the definitions and results for the set of non–null invariants for the case of $AdS_2 \times S^{d-2}$ geometry in $d = 4$ and $d = 5$. It is important to point out that for $d = 5$ it is not possible to calculate the complex Riemann invariants, as they are yet unknown. Additional invariants may be required for a more general spacetime like the $d = 5$ case. The question of how many invariants are necessary to obtain a *complete set*, for $d > 4$ is an open problem. Likewise, the invariants in Table 2 are organized by their respective degree. Following the definitions given in [21], the $j - th$ invariant I^p_j is called the invariant of *order* p, and then if other invariant exists that can be written in the form $I^p_j I^q_k I^r_l$, it is said to be an invariant of *order* $p - q - r$ and the sum $p + q + r$ is his *degree*. For example, in Table 1 the invariant $\mathfrak{Re}(m_1)$, which denotes the real part of the invariant m_1, is of *order* $1 - 2$ and has third-*degree*. Thus, the invariant I^p_j will be *independent* if it can not be written in terms of other invariants, either of equal or lower degree. Also, two invariants are said to be *equivalent* if they can be written in terms of each other, or as the product of other invariants of lower degree. The *equivalence* relations (syzygies) are also given in Table-2. It is easy to see that in $d = 4$ and $d = 5$ all the invariants can be written in terms of the R, r_1 and r_2 invariants.

It is important to remark that all the non-zero invariants are taken into account in our calculations. The remaining null invariants are not included in the Table 1-2. Naturally, an important issue may arise in these tables, namely: why there are no invariants or higher order terms of a gauge theory given the fact that we have started from an Einstein-Maxwell theory where a gauge is contained? The answer to this question is that, for simplicity, we have not considered in this paper terms such as $(F_{\mu\nu}F^{\mu\nu})^2$, $F^{\mu}_{\ \nu}F^{\nu}_{\ \rho}F^{\rho}_{\ \sigma}F^{\sigma}_{\ \mu}$, $R\,F^2$,..., nor covariant $F_{\mu\nu}\Box F^{\mu\nu}$,..., nor invariants of forms. Hence, just the purely gravitational sector of the theory is considered in the CM set. All calculations were carried out with the tensor package GRTensor running on the algebraic computing program Maple.

This paper is organized in six sections. In section II the generalized theory for $d = 4$ and $d = 5$ is written with higher derivative terms built from the set of Riemann invariants. The analytic solution in $d = 4$ for invariants of second degree is obtained in section III. Approximate solutions for $d = 4$ of the complete set of Riemann invariants are also examined in section III. Analytic and approximate solutions are provided in section IV for $d = 5$. The generic GB case is shortly treated in section V. We present our conclusion in section VI, while the Appendix provides the set of Riemann invariants for an extreme BH background.

2. Generalized theory in 4 and 5 dimensions.

We consider in this section a higher order theory of gravity by introducing the complete set of non–null Riemann invariants as the higher derivative terms of the theory. We construct

then the Einstein–Maxwell action with cosmological constant and the additional higher derivative terms like,

$$\mathcal{S} = \frac{1}{16\pi G_d}\int dx^d\sqrt{-g}\left(R+\Lambda-\frac{F^2}{4}+\mathcal{L}^d_{inv}\right), \tag{2.1}$$

where:

$$\begin{aligned}
\mathcal{L}^{d=4}_{inv} &= a_2R^2+b_2R_2+a_3R^3+b_3R\,R_2+a_4R^4+b_4R_2^2+c_4R^2\,R_2+a_5R^5+b_5R^3\,R_2\,,\\
\mathcal{L}^{d=5}_{inv} &= a_2R^2+b_2R_2+a_3R^3+b_3R\,R_2+c_3R_3+a_4R^4+b_4R_2^2+c_4R^2\,R_2+e_4R\,R_3\,.
\end{aligned}$$

with G_d being the d–dimensional Newton constant, R the Ricci scalar, Λ the cosmological constant, $F_{\mu\nu}$ the electromagnetic tensor, and $F^2 = F_{\mu\nu}F^{\mu\nu}$, R_2 and R_3 the two first real Riemann invariants defined in Table 1 for the metric (2.2). The parameters a_i, b_i, c_i, e_i... are the coupling constants for each higher derivative term of i–th degree. The label $\mathcal{L}^d_{inv}$ denotes the higher derivative terms with which we will shall work. In both cases $d = 4$ and $d = 5$ the invariant terms inside $\mathcal{L}^d_{inv}$ form a complete set, therefore we added all possible higher derivative terms until the highest degree [1]. The most general spacetime for a static and extremal BH with $AdS_2\times S^{d-2}$ topology near the horizon of geometry is:

$$ds^2 = v_1(-r^2dt^2+\frac{dr^2}{r^2})+v_2d\Omega_{d-2}, \tag{2.2}$$

$$e^{a_I\Psi_I}|_H = u_I,\;\; F^d_{0r}=e,\;\; F^{d=4}_{\theta\phi}=p\sin\theta,\;\; F^{d=5}_{\theta\phi}=0, \tag{2.3}$$

$$d\Omega^2_{d-2} = d\theta_1^2+\sum_{i=2}^{d-2}\prod_{j=1}^{i-1}\sin^2\theta_jd\theta_i^2,\;\; 0\le\theta_i\le\pi,\, 0\le\theta_{d-2}\le 2\pi,\, (1\le i\le d-3). \tag{2.4}$$

where e and p are functions related to the electric and magnetic charges, while the v_1 and v_2 are functions connected with the BH throat. The constants u_i are the values of the scalars fields Ψ_i on the BH horizon, if these fields are present (we assume $u_i = 0$). We shall follow in this paper the formalism of Sen [1], in which the entropy function $\mathcal{E}$ is defined as:

$$\mathcal{E}(\vec{u},\vec{v},\vec{e},\vec{q},\vec{p}) = 2\pi(e_iq_i - f(\vec{u},\vec{v},\vec{e},\vec{p})). \tag{2.5}$$

and $f(\vec{u},\vec{v},\vec{e},\vec{p})$ is the Lagrangian density $\sqrt{-det\,g}\mathcal{L}$, evaluated near the horizon of this geometry. All these parameters can be determined by extremizing the entropy function:

$$\frac{\partial\mathcal{E}}{\partial u_i}=\frac{\partial\mathcal{E}}{\partial v_j}=\frac{\partial\mathcal{E}}{\partial e}=0,\qquad i=1..N, j=1..2. \tag{2.6}$$

The last set of equations are the equations of motion near the horizon of the extremal background (2.2). Thus, the BH entropy at the extremal limit follows after solving the system of equations (2.6) and substituting these parameters in the entropy function. The result shows that $S_{BH}/2\pi$ may be regarded as the Legendre transform of the function $f(\overrightarrow{u},\overrightarrow{v},\overrightarrow{e},\overrightarrow{p})$, with respect to the variables e_i. In the following section the second order contributions are obtained.

[1] By "the highest degree" we refer to the highest degree within the complete set of Carminati-McLenagham invariants inside the sector of pure gravity.

3. Case d=4

3.1 Analytic Solutions for invariants of second degree in $d = 4$.

The calculation of second order contributions to the extremal BH entropy is equivalent to considering invariants of second degree. On the basis of the correspondence principle, the Gauss-Bonnet(GB) solutions should be contained in these results [14], as well as the classic Reissner-Nordstrom (RN) solution that was obtained through the area law. In this case, we should take $a_i = b_i = c_i = 0$ for $i \geq 3$ in (2.1), and then the function $f(\vec{v}, \vec{e}, \vec{p})$ will be,

$$f(v_1, v_2, e, p) = \int_{S^2} \sqrt{-g}\left(R + \Lambda - \frac{F^2}{4} + a_2 R^2 + b_2 R_2\right) d\theta d\phi. \tag{3.1}$$

while the entropy function will take the form:

$$\begin{aligned}\mathcal{E}(v_1, v_2, e, q, p) \;=\; & \left\{ 2\,qeG_4 + 2\left(2 - \frac{v_1}{v_2} - \frac{v_2}{v_1}\right) a_2 + \left(-\frac{v_1}{8v_2} - \frac{v_2}{8v_1} - \frac{1}{4}\right) b_2 + \right. \\ & \left. + \left(\frac{p^2}{2v_2} - 2 - v_2\,\Lambda\right)\frac{v_1}{2} - \frac{v_2\,e^2}{4v_1} + v_2 \right\} \frac{\pi}{G_4}\,.\end{aligned} \tag{3.2}$$

From the equations of motion (2.6) we obtain the system:

$$2\,q - \frac{v_2\,e}{2v_1 G_4} = 0, \tag{3.3}$$

$$\frac{v_1\Lambda}{2} - 1 + 2\,\frac{a_2}{v_1} - 2\,\frac{v_1\,a_2}{{v_2}^2} + \frac{b_2}{8v_1} - \frac{v_1\,b_2}{8{v_2}^2} + \frac{e^2}{4v_1} + \frac{v_1\,p^2}{4{v_2}^2} = 0, \tag{3.4}$$

$$-v_2\,\frac{\Lambda}{2} - 1 + 2\,\frac{v_2\,a_2}{{v_1}^2} - 2\,\frac{a_2}{v_2} + \frac{v_2\,b_2}{8{v_1}^2} - \frac{b_2}{8v_2} + \frac{v_2\,e^2}{4{v_1}^2} + \frac{p^2}{4v_2} = 0. \tag{3.5}$$

Notice that when we solve the system of equations above, all its solutions can be written in terms of the function v_2 as follows,

$$\begin{aligned} v_1 &= \frac{v_2}{v_2\,\Lambda + 1}, \qquad q = \frac{f}{8G_4}, \qquad e = \frac{f}{2(v_2\,\Lambda + 1)}, \\ f &= \sqrt{8\,(\Lambda v_2 + 2)v_2 - 2(16\,a_2 + b_2)\,(v_2\Lambda + 2)v_2\Lambda - 4\,p^2}\,. \end{aligned} \tag{3.6}$$

and the entropy for an extremal and static BH (taking into account invariants of second degree) will be:

$$S_{BH} = \left(1 - \frac{(16\,a_2 + b_2)\,\Lambda}{4}\right)\frac{\pi v_2}{G_4} - \frac{\pi\,b_2}{2G_4}. \tag{3.7}$$

The GB solution that was found by Morales and Samtleben in [14] can then be obtained substituting $b_2 = -8\alpha$, and $a_2 = \alpha/2$ in (3.7), thus:

$$S_{GB} = (v_2 + 4\alpha)\frac{\pi}{G_4}. \tag{3.8}$$

where the parameter α is the GB coupling constant. Similarly, if we take $a_2 = b_2 = 0$ in (3.7), the entropy becomes the well-known RN black hole entropy in the extremal limit and with magnetic charge p:

$$S_{BH}|_{a_2=b_2=0} = \frac{\pi v_2}{G_4} \equiv S_{RN}^{d=4}. \tag{3.9}$$

The variables v_1, e and q become,

$$v_1 = \frac{v_2}{v_2\,\Lambda + 1}, \qquad q = \frac{\sqrt{2\,(\Lambda v_2 + 2)v_2 - \,p^2}}{4G_4}, \qquad e = \frac{\sqrt{2\,(\Lambda v_2 + 2)v_2 - \,p^2}}{v_2\,\Lambda + 1}. \tag{3.10}$$

where it is straightforward to see (from (3.7)) that the cosmological constant by itself does not change the Bekestein-Hawking area law, since it needs to be accompanied by higher derivative terms of at least of second degree. In fact, the cosmological constant Λ just changes the geometry of the throat (see (3.6),(3.10)). However, the constants a_2 and b_2 in (3.7) represent the deviation from this law. In the following section approximated solutions for higher order gravity are obtained, as well as the R^3, R^4 and R^5 contributions.

3.2 Approximated Solutions for the complete set of invariants in $d = 4$.

Due to the non-linearity of Einstein–Maxwell equations it is very difficult to find exact analytic solutions with higher derivative terms. In most cases, some approximation methods must be employed or solutions must be found numerically. If we consider the complete set of invariants, we cannot find the solutions of the system of motion equations (2.6) in explicit form. In order to solve this problem we shall introduce the parameter w and make suitable expansions around it. Indeed, w in (2.1) can always be extracted from the coupling constants a_i, b_i, c_i, e_i by rescaling. Though, we should take into account that w must be extracted with the appropriate order. Since RN solutions (case $\mathcal{L}_{inv} = 0$) are well known analytically, the first coupling constants for invariants of higher degree are a_2 and b_2. As a consequence: $a_2, b_2 \Rightarrow w\,a_2, w\,b_2, ...$ and in general, we have:

$$a_l, b_m, c_n, e_p \Longrightarrow w^{l-1}a_l, w^{m-1}b_m, w^{n-1}c_n, w^{p-1}e_p, \qquad l, m, n, p = 2, 3, 4, 5, ... \tag{3.11}$$

The expansion parameter w can be considered as the inverse function of the charge, which is zero to leading order. This is the form in which w should appear in the entropy function $\mathcal{E}$. Since we are mainly interested in approximate solutions, then we should build series expansions of the functions e, q and v_1. In general, any of these functions can be expanded in series around the parameter w as:

$$h(w) = \sum_{k=0}^{\infty} \frac{w^k}{k!} \left(\frac{\partial^k h}{\partial w^k}\right)_{w=0}. \tag{3.12}$$

This expansion allows us to write:

$$\begin{aligned} e &\simeq e_0 + w\Delta e_1 + w^2\Delta e_2 + w^3\Delta e_3 + ..., \\ q &\simeq q_0 + w\Delta q_1 + w^2\Delta q_2 + w^3\Delta q_3 + ..., \\ v_1 &\simeq v_{1\,0} + w\Delta v_{1\,1} + w^2\Delta v_{1\,2} + w^3\Delta v_{1\,3} + ..., \\ \mathcal{E} &\simeq \mathcal{E}_0 + w\Delta\mathcal{E}_1 + w^2\Delta\mathcal{E}_2 + w^3\Delta\mathcal{E}_3 + ... \end{aligned} \tag{3.13}$$

where in (3.13) we have taken $\Delta h_k = (1/k!)(\partial^k h/\partial w^k)_{w=0}$. Consequently, w fixes the level of approximation, so that when $w = 0$ we have solutions of order zero that will be the RN set (3.9)-(3.10). Finally, by including the complete set of invariants (i.e. in (2.1) all terms are taken completely), or the contributions of higher derivative terms up to fifth degree for extremal BH entropy, the approximation at different levels will be:

$$S_{BH} = S_{RN}^{d=4} + \left(S_{BH}^{(2)} + S_{BH}^{(3)} w + S_{BH}^{(4)} w^2 + S_{BH}^{(5)} w^3\right) w + O(w^5), \tag{3.14}$$

where:

$$S_{BH}^{(2)} = -\left((16\,a_2 + b_2)\,\frac{\Lambda}{4} v_2 + \frac{b_2}{2}\right)\frac{\pi}{G_4}, \tag{3.15}$$

$$S_{BH}^{(3)} = \left(12\Lambda^2 v_2\, a_3 + (\frac{3}{4} v_2 \Lambda^2 + 2\Lambda + \frac{1}{v_2}) b_3\right)\frac{\pi}{G_4}, \tag{3.16}$$

$$\begin{aligned} S_{BH}^{(4)} = &-\left\{32\, v_2\, \Lambda^3\, a_4 + \frac{(v_2\,\Lambda + 2)^3\; b_4}{8 {v_2}^2} + \frac{2\Lambda\;(v_2\,\Lambda + 2)\,(v_2\,\Lambda + 1)\; c_4}{v_2} + (16 a_2 + b_2) \times \right. \\ &\times \left. \left(\frac{\Lambda\,(v_2 \Lambda + 2)^2\, b_3}{16 v_2} + v_2 a_3 \Lambda^3\right)\right\}\frac{\pi}{G_4}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} S_{BH}^{(5)} = &\left\{80 v_2 \Lambda^2 a_5 + (5 v_2 \Lambda^2 + 16\Lambda + \frac{12}{v_2}) b_5 + 96 v_2 \Lambda^2 a_3^2 + 4(\frac{6}{v_2} + 3 v_2 \Lambda^2 + 8\Lambda) a_3 b_3 + \right. \\ &+ \left. \frac{(3 v_2 \Lambda + 4)(v_2 \Lambda + 2)^2}{8 \Lambda v_2^2} b_3^2 + (16 a_2 + b_2)\left(4\Lambda^2 a_4 v_2 + \frac{(v_2 \Lambda + 2)^2 c_4}{4 v_2} + \frac{(v_2 \Lambda + 2)^4 b_4}{64 \Lambda^2 v_2^3}\right)\right\}\frac{\pi\,\Lambda^2}{G_4}. \end{aligned} \tag{3.18}$$

If we consider an effective theory where just invariants of third degree are required, then we should take in (2.1), (3.2) the constants with values $a_j = b_j = c_j = e_j = 0$ for $j \geq 4$, and then the entropy solutions for extremal BH at a linear approximation is simply: $S_{BH} = S_{RN} + (S_{BH}^{(2)} + S_{BH}^{(3)} w)w + O(w^3)$. The same can be done for fourth degree. We can consider this approach more than enough because the analytical solution of second degree (3.7) is exactly reproduced in (3.15). Note that each entropy contribution of invariants of $i - th$ degree is labeled as $S^{(i)}$, then the super-indices (i) just indicate the degree of the invariants that produced it and not the approximation order. Note that we have not written the solutions for e, q and v_1, because they are too long to be written in a paper. In the set of approximated solutions the non-linear terms with respect to the coupling constants appear in $S_{BH}^{(4)}$ and $S_{BH}^{(5)}$ with contributions like $a_2 a_3, a_2 b_3, b_2 a_3, b_2 b_3$ and $a_3^2, a_3 b_3, b_3^2, a_2 a_4, a_2 c_4, a_2 b_4, b_2 a_4, b_2 c_4, b_2 b_4$ respectively.

4. Case d=5

4.1 Analytic Solutions for invariants of second degree in $d = 5$.

In this case, the function $f(\vec{v}, \vec{e}, \vec{p})$ takes the same form as (3.1), but the integration is carried on over S^3, hence the entropy function is given by:

$$\mathcal{E}(v_1, v_2, e, q, p) = 2\,\pi\, qe - \frac{\pi^2\,(3\,v_1 - v_2)^2\, a_2}{G_5\, v_1\,\sqrt{v_2}} - \frac{3}{40}\,\frac{\pi^2\,(2\,v_1 + v_2)^2\, b_2}{G_5\, v_1\,\sqrt{v_2}} -,$$

$$- \frac{\sqrt{v_2}\left(2v_1v_2(\Lambda v_1 - 2) + e^2 v_2 + 12\, {v_1}^2\right)\pi^2}{8G_5\, v_1}. \tag{4.1}$$

Since the system of equations of motion (2.6) near the horizon are:

$$\begin{aligned}
0 &= \frac{\pi\,(3v_1 + v_2)\,(3v_1 - v_2)\,a_2}{\sqrt{v_2}{v_1}^2} + \frac{3\pi\,(2\,v_1 - v_2)\,(2\,v_1 + v_2)\,b_2}{40\sqrt{v_2}{v_1}^2} - \frac{\pi\,\sqrt{v_2}\left(-12\,{v_1}^2 - 2\,\Lambda\,{v_1}^2 v_2 + e^2 v_2\right)}{8{v_1}^2},\\
0 &= \frac{3\pi\,(v_1 + v_2)\,(3v_1 - v_2)\,a_2}{\sqrt{v_2}{v_1}^2} + \frac{3\pi\,(2\,v_1 + v_2)\,(2\,v_1 - 3\,v_2)\,b_2}{40\sqrt{v_2}{v_1}^2} - \frac{3\pi\,\sqrt{v_2}\left(2v_1^2(2 + \Lambda v_2) + v_2(e^2 - 4v_1)\right)}{8{v_1}^2},\\
0 &= q - \frac{\pi\,{v_2}^{\frac{3}{2}} e}{v_1\, 8G_5},
\end{aligned} \tag{4.2}$$

this system can be solved explicitly, and then the functions e, v_1 and q can be written in terms of v_2 as:

$$v_1 = \frac{v_2\,(5\,v_2 + 20a_2 - b_2)}{60a_2 + 2\,b_2 + 5v_2(4 + v_2\Lambda)}, \tag{4.3}$$

$$q = \frac{\pi\,\sqrt{v_2}\,\widetilde{f}}{8\,(5v_2 + 20a_2 - b_2)\,G_5}, \quad e = \frac{\widetilde{f}}{20v_2 + 60a_2 + 2b_2 + 5v_2^2\Lambda}. \tag{4.4}$$

and,

$$\widetilde{f} = \sqrt{5}\left({v_2}^2\Lambda + 20a_2 + 6v_2\right)^{\frac{1}{2}}\left(10v_2^2 - 60a_2b_2 - 40a_2v_2^2\Lambda - 2b_2^2 - 3b_2v_2^2\Lambda - 10b_2v_2\right)^{\frac{1}{2}}.$$

Therefore, the entropy of a static and extremal BH in $d = 5$, with higher derivative terms of second degree taken into account, has the form:

$$S_{BH} = \frac{\pi^2 {v_2}^{\frac{5}{2}}\,(40a_2 + 3b_2)\,\Lambda}{4\,(-5v_2 - 20\,a_2 + b_2)\,G_5} + \frac{5\pi^2 v_2^{\frac{1}{2}}\left(-{v_2}^2 + 10a_2b_2 + 2b_2v_2\right)}{2\,(-5v_2 - 20a_2 + b_2)\,G_5}. \tag{4.5}$$

Likewise, the extremal Gauss-Bonnet solution shown in [14] can be obtained with the substitutions: $b_2 = -8\,\alpha$, and $a_2 = \frac{3}{5}\alpha$, in the solution (4.5), leading to:

$$S_{GB} = (v_2 + 12\alpha)\frac{\pi^2 {v_2}^{\frac{1}{2}}}{2G_5}. \tag{4.6}$$

This does not happen for a more general GB contribution (see next section). In general, establishing a relation between the GB's solutions and the solutions of second degree requires a constraint.

If $a_2 = b_2 = 0$ in (4.5), the well-known extremal RN solution is also obtained:

$$S_{BH}|_{a_2=b_2=0} = \frac{\pi^2 v_2^{\frac{3}{2}}}{2G_5} \equiv S_{RN}^{d=5}. \tag{4.7}$$

As with the solution (3.7), the cosmological constant does not change the entropy of a BH by itself: it would need the higher derivative terms. A proof of this follows from taking $\Lambda = 0$ in (4.5), so that the contributions of a_2 and b_2 remain. These contributions also produce a marked deviation from the area law. Note that, contrary to the $d = 4$ case, not only the cosmological constant modifies the geometry of the BH throat (see (4.3)), but the coupling constants associated to the invariants of second degree increase its effect on the throat topology.

4.2 Approximated Solutions for the complete set of invariants in $d = 5$.

As in section 3.2, we can construct approximated solutions in five dimensions. Thus, considering the extremal BH entropy in which the complete set of Riemann invariants (i.e. in (2.2) has been taken into account, the coupling constant $a_j, b_j, c_j, e_j \neq 0$) at different levels of approximation take the form:

$$S_{BH} = S_{RN}^{d=5} + (S_{BH}^{(2)} + S_{BH}^{(3)} w + S_{BH}^{(4)} w^2) w + O(w^4). \tag{4.8}$$

where:

$$S_{BH}^{(2)} = -\left(2\,(1+\Lambda\, v_2)\, a_2 + \frac{3}{20}\,(6+\Lambda\, v_2)\, b_2\right) \frac{\pi^2 v_2^{\frac{1}{2}}}{G_5}\ , \tag{4.9}$$

$$\begin{aligned} S_{BH}^{(3)} = & \Big\{ 6\,(1+\Lambda\, v_2)^2\, a_3 + \frac{3}{20}\,(6+\Lambda\, v_2)\,(3\,\Lambda\, v_2+8)\, b_3 - \frac{9}{400}\,(6+\Lambda\, v_2)^2\, c_3 + \\ & + 8(1+\Lambda v_2)a_2^2 + (-9+v_2\Lambda)\frac{a_2 b_2}{5} - \frac{3}{100}(v_2\Lambda+6)b_2^2 \Big\} \frac{\pi^2}{v_2^{\frac{1}{2}} G_5}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} S_{BH}^{(4)} = & -\Big\{ 16\,(1+\Lambda\, v_2)^3\, a_4 + \frac{9}{100}\,(6+\Lambda\, v_2)^3\, b_4 + \frac{3}{5}\,(6+\Lambda\, v_2)\,(2\,\Lambda\, v_2+7)\,(1+\Lambda\, v_2)\, c_4 - \\ & - \frac{3}{200}\,(4\,\Lambda\, v_2+9)\,(6+\Lambda v_2)^2\, e_4 - 32(1+v_2\Lambda)a_2^3 + \frac{4}{5}(11+v_2\Lambda)b_2 a_2^2 + \\ & + (9+4v_2\Lambda)\frac{a_2 b_2^2}{25} - 8(10+v_2\Lambda)(1+v_2\Lambda)^2 a_2 a_3 - (270+248 v_2\Lambda + 3v_2^3\Lambda^3 + 56 v_2^2\Lambda^2)\frac{a_2 b_3}{5} + \\ & + \frac{3}{100}(v_2\Lambda+6)(v_2^2\Lambda^2 + 16 v_2\Lambda + 30)a_2 c_3 - \frac{3}{500}(v_2\Lambda+6)b_2^3 - \frac{3}{5}(1+v_2\Lambda)(v_2\Lambda+5)\times \\ & \times (v_2\Lambda-4)a_3 b_2 - \frac{3}{200}(v_2\Lambda+6)(3v_2^2\Lambda^2+8v_2\Lambda-20)b_2 b_3 + \frac{9\Lambda v_2}{4000}(v_2\Lambda+6)^2 b_2 c_3 \Big\} \frac{\pi^2}{G_5 {v_2}^{\frac{3}{2}}}\,. \end{aligned} \tag{4.11}$$

It is straightforward to notice that if equation (4.5) is expanded in series up to second order of v_2^{-1}, then equation (4.9) is reproduced. If we take $a_2 = \frac{3}{5}\alpha$ and $b_2 = -8\alpha$ in (4.9) the GB solution (4.6) is obtained. In this case, the non-linear contributions of the coupling constants appear in $S_{BH}^{(3)}$ and $S_{BH}^{(4)}$ as combinations of coupling constants of smaller order like a_2^2, a_2b_2, b_2^2, and a_2^3,...etc. Note that the absent terms a_2^2, a_2b_2 and b_2^2 in 3.16 in contrast with 4.10 is simply a result of the calculation. The same happen in 3.17 and 4.11 with the terms, $a_2^3, b_2a_2^2, a_2b_2^2, b_2^3$.

5. Riemann invariants and generic Gauss-Bonnet gravity.

In this section we examine the case of Einstein–Maxwell theory with a slightly more general Gauss-Bonnet gravity (*i.e.* with three different coupling constants). The action of this theory can be written in d-dimensions as:

$$\mathcal{S} = \frac{1}{16\pi G_d}\int dx^d \sqrt{-g}\left(R + \Lambda - \frac{F^2}{4} + \chi_1 R_{\alpha\beta\gamma\delta}\, R^{\alpha\beta\gamma\delta} - 4\,\chi_2 R_{\mu\nu}\, R^{\mu\nu} + \chi_3 R^2\right). \tag{5.1}$$

so that, when the entropy function mechanism is applied, the entropy of the extremal BH for $d = 4$ takes the form:

$$S_{GB}^{(d=4)} = -\frac{2\pi\,\left({v_2}^2\Lambda^2 + 2\,v_2\,\Lambda + 2\right)\chi_1}{G_4\,\left(v_2\,\Lambda + 1\right)} + \frac{4\pi\,\left(2 + {v_2}^2\Lambda^2 + 2\,v_2\,\Lambda\right)\chi_2}{G_4\,\left(v_2\,\Lambda + 1\right)}, \tag{5.2}$$
$$- \frac{2\pi\,{v_2}^2\Lambda^2\chi_3}{G_4\,\left(v_2\,\Lambda + 1\right)} + \frac{\pi\,v_2}{G_4}\,.$$

The solutions for the functions v_1, e, and q are:

$$v_1 = \frac{v_2}{v_2\,\Lambda + 1}, \qquad e = \frac{f_0}{v_2\,\Lambda + 1}, \qquad q = -\frac{f_0}{4G_4}, \tag{5.3}$$
$$f_0 = (-8\chi_1\,{v_2}^2\Lambda^2 - 16\,\chi_1\,v_2\,\Lambda + 16\,\chi_2{v_2}^2\Lambda^2 + 32\,\chi_2\,v_2\Lambda - 8\chi_3{v_2}^2\Lambda^2 - 16\,\chi_3 v_2\Lambda + 2{v_2}^2\Lambda + 4v_2 - p^2)^{\frac{1}{2}}\,.$$

so that, if $\chi_1 = \chi_2 = \chi_3 = \chi$, in (5.2), the solutions (3.8) is obtained (with $\chi = \alpha$). However, if we want to find the general GB solutions (5.2) from the analytical solution (3.7)(or vice versa), then the following constraints should be applied on both results:

$$\chi_1 - 2\chi_2 + \chi_3 = 0, \qquad \chi_1 = 2\chi_2 + \frac{3b_2}{16} + a_2, \qquad \chi_3 = a_2 - \frac{b_2}{16}\,. \tag{5.4}$$

Likewise, for $d = 5$ the resultant entropy is given by,

$$S_{GB}^{(d=5)} = \frac{\pi^2}{(v_2 + 4\chi_3)G_5}(-8\sqrt{v_2}\chi_1^2 + \chi_1(48\chi_2\sqrt{v_2} - 2\sqrt{v_2}(4v_2 + 16\chi_3 + \Lambda\,v_2^2)) - 64\sqrt{v_2}\chi_2^2 +$$
$$+4\sqrt{v_2}(20\chi_3 + \Lambda v_2^2 + 4v_2)\chi_2 - \frac{v_2^{\frac{5}{2}}}{2}(4\chi_3\Lambda - 1))\,. \tag{5.5}$$

for the funtions v_1, e and q:

$$v_1 = \frac{v_2(v_2 + 4\chi_3)}{4v_2 + \Lambda v_2^2 - 16\chi_2 + 12\chi_3 + 4\chi_1}\,, \tag{5.6}$$
$$q = q(v_2, \Lambda, \chi_1, \chi_2, \chi_3), \qquad e = e(v_2, \Lambda, \chi_1, \chi_2, \chi_3)\,. \tag{5.7}$$

Also, the relation between the entropies (5.5) and (4.5) are given for the case when the following constraints hold:

$$\chi_1 - 2\chi_2 + \chi_3 = 0, \qquad a_2 = \frac{3}{5}(2\chi_2 - \chi_1), \qquad b_2 = 8(\chi_1 - 2\chi_2)\,. \tag{5.8}$$

If we take $\chi_1 = \chi_2 = \chi_3 = \chi$, in (5.5), equation (4.6) is obtained. These last results are interesting because they correspond to cases where the Riemann invariants cannot reproduce the results of the generic GB theory, and this suggests that this theory will always lack invariants to add. However, this situation can be understood if we notice that it is a well-known fact that generic GB gravity in AdS gravity is, in general, inconsistent by two essential points: the variational principle [26], and regularization problems, both subjects are discussed extensively in [27],[28],[29].

6. Conclusion.

We have calculated the entropy for extremal BHs in the cases $d = 4$ and $d = 5$, taking into account higher derivative terms built from the complete set of Riemann invariants, a task which, as far as we are aware, has not been accomplished before. Though we remark that the exceptional Gauss-Bonnet's cases were already done in [14]. We have found the invariants of second degree and generalized these results so that the GB case can be obtained as a particular case. We have also obtained the leading terms of approximation for higher order invariants that could be interpreted as contributions of higher derivative terms in some theories of gravity. The RN solutions are also contained in all the examined cases. Therefore, the use of our set of solutions provides a concrete example showing that the entropy function formalism works well, and its applications can be a less complicated process in comparison to the use of Wald's equation [22]-[25].

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Table 1: Table of definitions for the complete set of Riemann invariants and the results for the $AdS_2 \times S^{d-2}$ geometry near the BH horizon in $d = 4$ and $d = 5$ (the rest of the invariants are zero). All the definitions are based on the trace–free Ricci tensor $S_{a\,b}$, the Weyl tensor C_{abcd} and the Riemann tensor R_{abcd}. Although, these invariants can also be defined on a spinor base [20]. Here "-" means that there is no invariant definition available in five dimensions for the set of complex invariants. We use small letters for the invariants of Carminati [20], but in the Lagrangean we use upper case letters. The $\mathfrak{Re}$ symbol means real part of the complex invariant. In order to remark the tensor degree (instead of Carminati subindex) we have written the tensor degree number as a subindex (*i.e.* $\mathfrak{Re}(W_2)$ means the real part of the second degree complex Weyl invariant w_1).

$Invariants$	$Definitions$	$Solutions$ $d = 4$	$Solutions$ $d = 5$
R	$g^{ad}g^{bc}R_{abcd}$	$2\,\gamma_1$	$2\gamma_3$
$r_1 = R_2$	$\frac{1}{4}S_a{}^bS_b{}^a$	$\frac{1}{4}\gamma_2^2$	$\frac{3}{10}\gamma_4^2$
$r_2 = R_3$	$-\frac{1}{8}S_a{}^bS_b{}^cS_c{}^a$	0	$\frac{3}{10^2}\gamma_4^3$
$r_3 = R_4$	$\frac{1}{16}S_a{}^bS_b{}^cS_c{}^dS_d^a$	$\frac{1}{64}\gamma_2^4$	$\frac{21}{10^3}\gamma_4^4$
$\mathfrak{Re}(w_1) = \mathfrak{Re}(W_2)$	$\frac{1}{8}C_{abcd}C^{abcd}$	$\frac{1}{6}\gamma_1^2$	$-$
$\mathfrak{Re}(w_2) = \mathfrak{Re}(W_3)$	$-\frac{1}{16}C_{ab}^{\ \ cd}C_{cd}^{\ \ ef}C_{ef}^{\ \ ab}$	$\frac{-1}{36}\gamma_1^3$	$-$
$\mathfrak{Re}(m_1) = \mathfrak{Re}(M_3)$	$\frac{1}{8}S^{ab}S^{cd}C_{acdb}$	$\frac{-1}{12}\gamma_1\gamma_2^2$	$-$
$\mathfrak{Re}(m_2) = \mathfrak{Re}(M_4)$	$\frac{1}{16}S^{cd}S_{ef}(C_{acdb}C^{aefb} - C^*_{acdb}C^{*aefb})$	$\frac{1}{36}\gamma_1^2\gamma_2^2$	$-$
$m_3 = M_4^+$	$\frac{1}{16}S^{cd}S_{ef}(C_{acdb}C^{aefb} + C^*_{acdb}C^{*aefb})$	$\frac{1}{36}\gamma_1^2\gamma_2^2$	$-$
$\mathfrak{Re}(m_5) = \mathfrak{Re}(M_5)$	$\frac{1}{32}S^{cd}S^{ef}C^{aghb}(C_{acdb}C_{gefh} + C^*_{acdb}C^*_{gefh})$	$\frac{-1}{108}\gamma_1^3\gamma_2^2$	$-$
$where:$	$\gamma_1 = (v_1 - v_2)/v_1v_2,\ \gamma_2 = (v_1 + v_2)/v_1v_2,$ $\gamma_3 = (3v_1 - v_2)/v_1v_2,\ \gamma_4 = (2v_1 + v_2)/v_1v_2$		

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Table 2: The complete set of not null invariants is provided, organized by *degrees* for $d = 4$ and $d = 5$ dimensions on $AdS_2 \times S^{d-2}$ geometry.

d	*Degree*	*Invariants*	*Equivalence relations*
$d = 4$	$1^{\underline{st}}$	$R = 2\gamma_1$	γ_1
	$2^{\underline{nd}}$	$R^2 = 4\gamma_1^2, \quad R_2 = \frac{1}{4}\gamma_2^2, \quad \mathfrak{Re}(W_2) = \frac{1}{6}\gamma_1^2,$	$\gamma_1^2,\ \gamma_2^2$
	$3^{\underline{rd}}$	$R^3 = 8\gamma_1^3, \quad R \times R_2 = \frac{1}{2}\gamma_1\gamma_2^2, \quad R \times \mathfrak{Re}(W_2) = \frac{1}{3}\gamma_1^3,$ $\mathfrak{Re}(W_3) = \frac{-1}{36}\gamma_1^3, \quad \mathfrak{Re}(M_3) = \frac{-1}{12}\gamma_1\gamma_2^2$	$\gamma_1^3,\ \gamma_1\gamma_2^2$
	$4^{\underline{th}}$	$R^4 = 16\gamma_1^4, \quad \mathfrak{Re}^2(W_2) = \frac{\gamma_1^4}{36}, \quad R^2 \times R_2 = \gamma_1^2\gamma_2^2,$ $R_2^2 = \frac{1}{16}\gamma_2^4, \quad R^2 \times \mathfrak{Re}(W_2) = \frac{2}{3}\gamma_1^4,$ $R_2 \times \mathfrak{Re}(W_2) = \frac{1}{24}\gamma_1^2\gamma_2^2, \quad \mathfrak{Re}(M_4) = M_4^+ = \frac{1}{36}\gamma_1^2\gamma_2^2,$ $R \times \mathfrak{Re}(W_3) = \frac{-\gamma_1^4}{18}, \quad R \times \mathfrak{Re}(M_3) = \frac{-\gamma_1^2\gamma_2^2}{6},$ $R_4 = \frac{1}{64}\gamma_2^4,$	$\gamma_1^4,\ \gamma_2^4,\ \gamma_1^2\gamma_2^2$
	$5^{\underline{th}}$	$R^5 = 32\gamma_1^5, \quad R^3 \times R_2 = 2\gamma_1^3\gamma_2^2, \quad R^3 \times \mathfrak{Re}(W_2) = \frac{4}{3}\gamma_1^5,$ $R^2 \times \mathfrak{Re}(W_3) = -\frac{\gamma_1^5}{9}, \quad R_2^2 \times R = \frac{\gamma_1\gamma_2^4}{8},$ $R^2 \times \mathfrak{Re}(M_3) = -\frac{\gamma_1^3\gamma_2^2}{3}, \quad R \times \mathfrak{Re}^2(W_2) = \frac{\gamma_1^5}{18},$ $R \times R_2 \times \mathfrak{Re}(W_2) = \frac{\gamma_1^3\gamma_2^2}{12},$ $R_2 \times \mathfrak{Re}(W_3) = \frac{-\gamma_1^3\gamma_2^2}{144}, \quad R_2 \times \mathfrak{Re}(M_3) = \frac{-\gamma_1\gamma_2^4}{48},$ $\mathfrak{Re}(W_2) \times \mathfrak{Re}(M_3) = \frac{-\gamma_1^3\gamma_2^2}{72}, \quad \mathfrak{Re}(W_2) \times \mathfrak{Re}(W_3) = \frac{-\gamma_1^5}{216},$ $R \times R_4 = \frac{\gamma_1\gamma_2^4}{32}, \quad R \times \mathfrak{Re}(M_4) = R \times M_4^+ \equiv \frac{\gamma_1^3\gamma_2^2}{18}$	$\gamma_1^5, \gamma_1\gamma_2^4, \gamma_1^3\gamma_2^2$
$d = 5$	$1^{\underline{st}}$	$R = 2\gamma_3$	γ_3
	$2^{\underline{nd}}$	$R^2 = 4\gamma_3^2, \quad R_2 = \frac{3}{10}\gamma_4^2$	$\gamma_3^2,\ \gamma_4^2$
	$3^{\underline{rd}}$	$R^3 = 8\gamma_3^3, \quad R \times R_2 = \frac{3}{5}\gamma_3\gamma_4^2, \quad R_3 = \frac{3}{100}\gamma_4^3$	$\gamma_3^3,\ \gamma_3\gamma_4^2,\ \gamma_4^3$
	$4^{\underline{th}}$	$R^4 = 16\gamma_3^4, \quad R_2^2 = \frac{9}{100}\gamma_4^4, \quad R^2 \times R_2 = \frac{6}{5}\gamma_3^2\gamma_4^2,$ $R \times R_3 = \frac{3}{50}\gamma_3\gamma_4^3, \quad R_4 = \frac{21}{10^3}\gamma_4^4$	$\gamma_3^4,\ \gamma_4^4,\ \gamma_3^2\gamma_4^2,$ $\gamma_3\gamma_4^3$
		where : $\gamma_1 = (v_1 - v_2)/v_1v_2,\ \gamma_2 = (v_1 + v_2)/v_1v_2,$ $\gamma_3 = (3v_1 - v_2)/v_1v_2,\ \gamma_4 = (2v_1 + v_2)/v_1v_2$	

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